PRELIMINARIES

1. Conventions. The book is divided into 16 chapters, each subdivided into sections numbered in order (e.g. chapter 12, section 3 is numbered **12.3**).

Within each chapter results (Theorems, Propositions or Lemmas) are labelled by the chapter and then the order of occurrence (e.g. the fifth result in chapter 3 is `Proposition 3.5`). The exceptions to this rule are: sublemmas which are presented within the context of the proof of a more important result (e.g. the proof of Theorem 2.2 contains Sublemmas 2.2.1 and 2.2.2); and corollaries (the corollary to Theorem 5.5 is Corollary 5.5.1).

We denote the end of a proof by ■.

Finally, equations are numbered by the chapter and their order of occurrence (e.g. the fourth equation in chapter 5 is labelled (5.4))

2. Notation. We shall use the standard notation: $\mathbb{R}$ to denote the *real numbers*; $\mathbb{Q}$ to denote the *rational numbers*; $\mathbb{Z}$ to denote the *integer numbers*; $\mathbb{N}$ to denote the *natural numbers*; and $\mathbb{Z}^+$ to denote the non-negative integers. We use the convenient convention that: $\mathbb{R}/\mathbb{Z} = \{x + \mathbb{Z} : x \in \mathbb{R}\}$ (which is homeomorphic to the standard unit circle); $\mathbb{R}^2/\mathbb{Z}^2 = \{(x_1, x_2) + \mathbb{Z}^2 : (x_1, x_2) \in \mathbb{R}^2\}$ (which is homeomorphic to the standard 2-torus); etc. However, for $x \in \mathbb{R}$ we denote the corresponding element in $\mathbb{R}/\mathbb{Z}$ by $x \pmod 1$ (and similarly for $\mathbb{R}^2/\mathbb{Z}^2$, etc.).

We denote the interior of a subset A of a metric space by $\mathrm{int}(A)$, and we denote its closure by $\mathrm{cl}(A)$.

If $T : X \to X$ denotes a continuous map on a compact metric space then T^n $(n \geq 1)$ denotes the composition with itself n times.

If $T : I \to I$ is a C^1 map on the unit interval $I = [0, 1]$ then T' denotes its derivative.

3. Prerequisites in point set topology (chapters 1-6). The first six chapters consist of various results in topological dynamics for which the only prerequisite is a working knowledge of point set topology for metric spaces. For example:

THEOREM A (BAIRE). *Let X be a compact metric space; then if $\{U_n\}_{n\in\mathbb{N}}$ is a countable family of open dense sets then $\bigcap_{n\in\mathbb{N}} U_n \subset X$ is dense.*

THEOREM B (SEQUENTIAL COMPACTNESS). *Let X be a metric space; then X is compact if and only if every sequence $(x_n)_{n\in\mathbb{N}}$ in X contains a convergent subsequence.*

THEOREM C (ZORN'S LEMMA). *Let Z be a set with a partial ordering. If every totally ordered chain has a lower bound in Z then there is a minimal element in Z.*

Two good references for this material are [4] and [5]

4. Pre-requisites in measure theory (chapters 7-12). Chapters 7-12 form an introduction to ergodic theory, and suppose some familiarity (if not expertise) with abstract measure theory and harmonic analysis. The following results will be required.

THEOREM D (RIESZ REPRESENTATION). *There is a bijection between*

(1) *probability measures μ on a compact metric space X (with the Borel sigma algebra),*
(2) *Continuous linear functionals $c : C^0(X) \to \mathbb{R}$,*

given by $c(f) = \int f d\mu$.

THEOREM E. *Let $(X, \mathcal{B}, \mu)$ be a measure space. For every linear functional $\alpha : L^1(X, \mathcal{B}, \mu) \to L^1(X, \mathcal{B}, \mu)$ there exists $k \in L^\infty(X, \mathcal{B}, \mu)$ such that $\alpha(f) = \int f \cdot k d\mu$, $\forall f \in L^1(X, \mathcal{B}, \mu)$* [3, p.121].

In proving invariance of measures in examples the following basic result will sometimes be assumed.

THEOREM F (KOLMOGOROV EXTENSION). *Let $\mathcal{B}$ be the Borel sigma-algebra for a compact metric space X. If μ_1 and μ_2 are two measures for the Borel sigma-algebra which agree on the open sets of X then $m_1 = m_2$* [3, p. 310].

The following terminology will be used in the chapter on ergodic measures. Given two probability measures μ, ν we say that μ is *absolutely continuous* with respect to ν if for every set $B \in \mathcal{B}$ for which $\nu(B) = 0$ we have that $\mu(B) = 0$. We write $\mu << \nu$ and then we have the following result.

THEOREM G (RADON-NIKODYM). *If μ is absolutely continuous with respect to μ then there exists a (unique) function $f \in L^1(X, \mathcal{B}, d\nu)$ such that for any $A \in \mathcal{B}$ we can write $\mu(A) = \int_A f d\nu$.*

We usually write $f = \frac{d\mu}{d\nu}$ and call this the *Radon-Nikodym derivative* of μ with respect to ν.

We call two measures μ, ν *mutually singular* if there exists a set $B \in \mathcal{B}$ such that $\mu(A) = 0$ and $\nu(A) = 1$. We then write $\mu \perp \nu$.

In chapter 8 we shall need a passing reference to Lebesgue spaces. A *Lebesgue space* is a measure space which is measurably equivalent to the

the union of unit intervals (with the usual Lebesgue measure) with at most countably many points (with non-zero measure).

In chapter 11 we shall use the following result.

THEOREM H (DOMINATED CONVERGENCE). *Let* $h \in L^1(X, \mathcal{B}, \mu)$ *and let* $(f_n)_{n\in\mathbb{Z}^+}$
$\subset L^1(X, \mathcal{B}, \mu)$*, with* $|f_n(x)| \leq h(x)$*, converge (almost everywhere) to* $f(x)$*; then* $\int f_n d\mu \to \int f d\mu$ *as* $n \to +\infty$.

Good general references for this material are [1], [2], [3].

5. Subadditive sequences. A simple result which proves its worth several times in these notes is the following.

THEOREM F (SUBADDITIVE SEQUENCES). *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers such that* $a_{n+m} \leq a_n + a_m$*,* $\forall n, m \in \mathbb{N}$ *(i.e. a subadditive sequence); then* $a_n \to a$*, as* $n \to +\infty$*, where* $a = \inf\{a_n/n\colon n \geq 1\}$

PROOF. First note that $a_n \leq a_1 + a_{n-1} \leq \ldots \leq na_1$, and so $a \leq a_1$ For $\epsilon > 0$ we choose $N > 0$ with $a_N < N(a + \epsilon)$. For any $n \geq 1$ we can write $n = kN + r$, where $k \geq 0$ and $1 \leq r \leq N - 1$. Then

$$a_n \leq a_{kN} + a_r \leq ka_N + a_r \leq ka_N + \sup_{1\leq r\leq N} a_r$$

and we see that

$$\limsup_{n\to+\infty} \frac{a_n}{n} \leq \limsup_{k\to+\infty} \frac{ka_N + \sup_{1\leq r\leq N} a_r}{kN} = \frac{a_N}{N} \leq a + \epsilon.$$

This shows that $\frac{a_n}{n} \to a$, as required.

■

References

1. P. Halmos, *Measure Theory*, Van Nostrand, Princeton N.J., 1950.
2. K. Partasarathy, *An Introduction to Probability and Measure Theory*, Macmillan, New Delhi, 1977.
3. H. Roydon, *Real Analysis*, Macmillan, New York, 1968.
4. G. Simmons, *Introduction to Topology and Modern Analysis*, McGraw-Hill, New York, 1963.
5. W. Sutherland, *Introduction to Topological and Metric spaces*, Clarendon Press, Oxford, 1975.