CHAPTER 6

ROTATION NUMBERS

In this chapter we shall define the useful concept of the rotation number for orientation preserving homeomorphisms of the circle.

6.1 Homeomorphisms of the circle and rotation numbers

Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving homeomorphism of the circle to itself. There is a canonical projection $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ given by $\pi(x) = x \pmod 1$. We call a monotone map $\hat{T} : \mathbb{R} \to \mathbb{R}$ a *lift* of T if the canonical projection $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a semi-conjugacy (i.e. $\pi \circ \hat{T} = T \circ \pi$).

For a given map $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ a lift $\hat{T} : \mathbb{R} \to \mathbb{R}$ will not be unique.

EXAMPLE. If $T(x) = (x + \alpha) \pmod 1$ then for any $k \in \mathbb{Z}$ the map $\hat{T} : \mathbb{R} \to \mathbb{R}$ defined by $\hat{T}(x) = x + \alpha + k$ is a lift. To see this observe that $\pi(\hat{T}(x)) = \pi(x + \alpha + k) = x + \alpha \pmod 1$ and $T(\pi(x)) = \pi(x) + \alpha (\mathrm{mod}\ 1) = x + \alpha \pmod 1$.

The following lemma summarizes some simple properties of lifts.

LEMMA 6.1.

(i) *Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a homeomorphism of the circle; then if $\hat{T} : \mathbb{R} \to \mathbb{R}$ is a lift, then any other lift $\hat{T}' : \mathbb{R} \to \mathbb{R}$ must be of the form $\hat{T}'(x) = \hat{T}(x) + k$, for some $k \in \mathbb{Z}$.*

(ii) *For any $x, y \in \mathbb{R}$ with $|x - y| \le k$ ($k \in \mathbb{Z}^+$) we have $|\hat{T}(x) - \hat{T}(y)| \le k$. Iterating this gives that*

$$|\hat{T}^n(x) - \hat{T}^n(y)| \le k, \quad \forall n \ge 0.$$

PROOF. These are easily seen from the continuity and the monotonicity of $\hat{T}$.

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DEFINITION. We define the *rotation number* $\rho(T)$ of the homeomorphism by

$$\rho(T) = \limsup_{n \to +\infty} \frac{\hat{T}^n(x)}{n} \pmod 1.$$

(The limsup is independent of the choice $x \in \mathbb{R}$ by Lemma 6.1. The choice of lift $\hat{T}$ can only alter the limsup by an integer, which has no bearing since we define $\rho(T)$ modulo one.)

EXAMPLE. Consider the standard rotation $R_\rho : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by $R_\rho(x) = x + \rho \pmod 1$, where $\rho \in [0,1)$, say. Any lift $\hat{R}_\rho : \mathbb{R} \to \mathbb{R}$ will be of the form $\hat{R}_\rho(x) = x + \rho + k$, for some $k \in \mathbb{Z}$ i.e. translation on the real line by $\rho + k$. It is now immediate from the definition that the rotation number for R_ρ is merely $\rho \pmod 1$.

We can now show some interesting properties of $\rho(T)$.

PROPOSITION 6.2.

(i) *For $n \geq 1$ we have that $\rho(T^n) = n\rho(T) (\mathrm{mod}\ 1)$.*
(ii) *If T has a periodic point (i.e. $\exists n \geq 1, \exists x \in \mathbb{R}/\mathbb{Z}$ such that $T^n x = x$) then $\rho(T)$ is rational.*
(iii) *If $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has no periodic points then $\rho(T)$ is irrational.*
(iv) *The limit actually exists and we can write*

$$\rho(T) = \lim_{n \to +\infty} \frac{\hat{T}^n(x)}{n} (\mathrm{mod}\ 1).$$

PROOF. (i) Since $\hat{T}^n$ (the n th iterate of the lift $\hat{T}$ for T) is itself a lift for $T^n : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ this is immediate from the definitions.

(ii) Since $T^n(x+\mathbb{Z}) = x + \mathbb{Z}$ we have that $\hat{T}^n(x)$ and x differ by an integer (i.e. $\hat{T}^n(x) - x = k \in \mathbb{Z}$). Then for $pn + r$ with $0 \leq r \leq n-1$ and $p \geq 0$ we have that $\hat{T}^{pn+r}(x) = \hat{T}^r(\hat{T}^{pn}x) = \hat{T}^r(x) + pk$ because of Lemma 6.1. Thus $\rho(T) = \limsup_{p \to +\infty} \frac{\hat{T}^{pn+r}(x)}{pn+r} = \frac{k}{n} \pmod 1$.

(iii) Assume for contradiction that $\rho(T) = \frac{p}{q}$ is rational. By part (i) we see that for $S := T^q$ we have $\rho(S) = 0$ and since T has no periodic points we conclude that $S : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has no fixed points.

If $\hat{S} : \mathbb{R} \to \mathbb{R}$ is a lift for S then the absence of fixed points for S implies either $\hat{S}(x) > x$, $\forall x \in \mathbb{R}$, or $\hat{S}(x) < x$, $\forall x \in \mathbb{R}$. Assume $\hat{S}(x) > x$, $\forall x \in \mathbb{R}$ (the other case being similar), i.e. $\hat{S}$ is strictly increasing. If $\exists k > 0$ with $\hat{S}^k(0) > 1$ then we see that $\hat{S}^{mk}(0) > m$ and so

$$\rho(S) = \limsup_{n \to +\infty} \frac{\hat{S}^n(x)}{n} > \frac{1}{k},$$

contradicting that $\rho(S) = 0$. This leaves the possibility that $\hat{S}^k(0) < 1$ for all $k \geq 1$. Since $\hat{S}$ is strictly increasing the sequence $(\hat{S}^k(0))_{k=1}^{\infty}$ is monotone increasing and the supremum $z \in \mathbb{R}$ satisfies $\hat{S}(z) = z$. Thus S has a fixed point $S(z + \mathbb{Z}) = z + \mathbb{Z}$, giving a contradiction.

(iv) If T has a periodic point $T^n x = x$ then the argument in part (i) actually shows that $\rho(T) = \lim_{N\to+\infty} \frac{T^N(x)}{N} = \frac{k}{n}$ (mod 1), in particular, showing that the limit exists.

Assume that $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has no periodic points. Thus for all $n \geq 1$ there exists $k_n \in \mathbb{Z}$ such that $\hat{T}^n(x) - x \in [k_n, k_n + 1]$, $\forall x \in \mathbb{R}$, and, in particular, observe that $|\frac{\hat{T}^n(0)}{n} - \frac{k_n}{n}| < \frac{1}{n}$. Then, for any $m \geq 1$ we have that

$$\begin{aligned}\hat{T}^{nm}(0) = \hat{T}^n\left(\hat{T}^{n(m-1)}(0)\right) - \left(\hat{T}^{n(m-1)}(0)\right) \\ + \hat{T}^n\left(\hat{T}^{n(m-2)}(0)\right) - \left(\hat{T}^{n(m-2)}(0)\right) + \dots \\ \dots + \hat{T}^n\left(T^n(0)\right) - \left(\hat{T}^n(0)\right) + \hat{T}^n(0) \in [mk_n, m(k_n+1)].\end{aligned} \tag{6.1}$$

In particular, we see from (6.1) that $|\frac{\hat{T}^{nm}(0)}{nm} - \frac{k_n}{n}| < \frac{1}{n}$. The triangle inequality gives that

$$\begin{aligned}|\frac{\hat{T}^m(0)}{m} - \frac{\hat{T}^n(0)}{n}| &\leq |\frac{\hat{T}^m(0)}{m} - \frac{k_m}{m}| + |\frac{k_m}{m} - \frac{\hat{T}^{mn}(0)}{mn}| \\ &+ |\frac{\hat{T}^{mn}(0)}{mn} - \frac{k_n}{n}| + |\frac{k_n}{n} - \frac{\hat{T}^n(0)}{n}| \\ &\leq \frac{2}{m} + \frac{2}{n}\end{aligned}$$

which shows that the sequence $\left(\frac{\hat{T}^n(0)}{n}\right)_{n=0}^{\infty}$ is Cauchy, and in particular the limit exists.

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The next lemma and its corollary will be useful later.

LEMMA 6.3. *Assume that ρ is irrational.*

(i) *Let $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ and $x, y \in \mathbb{R}$. If $\hat{T}^{n_1}(x) + m_1 < \hat{T}^{n_2}(x) + m_2$ then $\hat{T}^{n_1}(y) + m_1 < \hat{T}^{n_2}(y) + m_2$;*

(ii) *The bijection $n\rho(T) + m \to \hat{T}^n(0) + m$ between the sets*

$$\Omega = \{n\rho(T) + m\colon\ n, m \in \mathbb{Z}\} \text{ and } \Lambda = \{\hat{T}^n(0) + m\colon\ n, m \in \mathbb{Z}\}$$

preserves the natural ordering on $\mathbb{R}$.

PROOF. (i) If $\exists x, y \in \mathbb{R}$ for which the ordering is reversed, then by continuity (and the intermediate value theorem) there exists $z \in \mathbb{R}$ with $\hat{T}^{n_1}(z) + m_1 = \hat{T}^{n_2}(z) + m_2$, i.e. $\hat{T}^{n_1}(z) - \hat{T}^{n_2}(z) \in \mathbb{Z}$. But then $T^{n_1 - n_2}(z + \mathbb{Z}) = z + \mathbb{Z}$ is a periodic point. This contradicts the assumption that T has irrational rotation number (and so no periodic points by Proposition 6.2 (ii)).

(ii) Assume that $\hat{T}^{n_1}(0) + m_1 < \hat{T}^{n_2}(0) + m_2$ with $n_1 > n_2$; then we wish to show that $n_1\rho + m_1 < n_2\rho + m_2$. We can rewrite the first inequality as $\hat{T}^{n_1-n_2}(\hat{T}^{n_2}0) - \hat{T}^{n_2}0 < (m_2 - m_1)$ and we can apply part (i) with $x = \hat{T}^{n_2}0$ and $y = 0$ to deduce that

$$\hat{T}^{n_1-n_2}(0) < (m_2 - m_1). \tag{6.2}$$

Next, we can apply part (i) to (6.2) with the choices $x = \hat{T}^{n_1-n_2}(0)$ and $y = 0$ to deduce

$$\begin{aligned} \hat{T}^{2(n_1-n_2)}(0) - \hat{T}^{n_1-n_2}(0) &= \hat{T}^{n_1-n_2}(\hat{T}^{n_1-n_2}0) - \hat{T}^{n_1-n_2}(0) \\ &< (m_2 - m_1). \end{aligned} \tag{6.3}$$

Comparing (6.2) and (6.3) gives that $\hat{T}^{2(n_1-n_2)}(0) < 2(m_2 - m_1)$. Proceeding inductively shows that for any $N \geq 1$ we have $\hat{T}^{N(n_1-n_2)}(0) < N(m_2 - m_1)$. Finally, we see that

$$\rho(T) = \lim_{n\to+\infty} \frac{\hat{T}^n(0)}{n} = \lim_{N\to+\infty} \frac{\hat{T}^{N(n_1-n_2)}(0)}{N(n_1 - n_2)} \leq \frac{(m_2 - m_1)}{(n_1 - n_2)}$$

and in fact $\rho < \frac{(m_2-m_1)}{(n_1-n_2)}$ since ρ is assumed irrational. This is the required inequality, and so this completes the proof.

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COROLLARY 6.3.1. *Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ have irrational rotation number ρ. For any $x \in \mathbb{R}/\mathbb{Z}$ the orbits of x under T and the rotation $R_\rho : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ have the same ordering.*

PROOF. This follows immediately from part (ii) of Lemma 6.3, since a difference in the ordering of $T^n(x)$ and $R_\rho^n(x)$ would contradict the conclusion of the lemma.

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6.2 Denjoy's theorem

The following result gives a sufficient condition for a homeomorphism to be conjugate to a rotation.

PROPOSITION 6.4. *If $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a minimal orientation preserving homeomorphism with irrational rotation number ρ then T is topologically conjugate to the standard rotation $R_\rho : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.*

PROOF. Let $\hat{T} : \mathbb{R} \to \mathbb{R}$ be a lift of $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. Observe that since ρ is irrational we have that $\Omega \subset \mathbb{R}$ (as defined in Lemma 6.3 (ii)) is dense. Moreover, since T is minimal we know that $\{T^n 0\}$ is dense in $\mathbb{R}/\mathbb{Z}$, and so we also have that $\Lambda \subset \mathbb{R}$ is dense.

The map $\phi : \Lambda \to \Omega$ given by $\phi(\hat{T}^n(0) + m) = n\rho + m$ is order preserving by Lemma 6.3. Thus it extends to a homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$.

Observe that

$$\phi\left(\hat{T}(\hat{T}^n(0) + m)\right) = \phi\left(\hat{T}^{n+1}(0) + m)\right) = (n+1)\rho + m$$

and

$$\hat{R}_\rho \phi(\hat{T}^n(0) + m) = \hat{R}_\rho(n\rho + m) = (n+1)\rho + m$$

where $\hat{R}_\rho : \mathbb{R} \to \mathbb{R}$ is a lift for R_ρ. Thus $\phi \circ \hat{T} = \hat{R}_\rho \circ \phi$.

Finally, we observe that by construction $\phi(x+1) = \phi(x) + 1$. Thus the homeomorpism $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by $\phi(x + \mathbb{Z}) = \phi(x) + \mathbb{Z}$ is well-defined. Moreover, the identity $\phi \circ \hat{T} = \hat{R}_\rho \circ \phi$ implies the conjugacy relation $\phi \circ T = R_\rho \circ \phi$. This completes the proof of the proposition.

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Given a C^1 map $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ we consider its derivative $T' : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$.

DEFINITION. We define the *variation* of $\log |T'| : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ by

$$\begin{aligned}&\mathrm{Var}(\log |T'|)\\ &= \sup \left\{ \sum_{i=0}^{n-1} |\log |T'|(x_{i+1}) - \log |T'|(x_i)| \, : \, 0 = x_0 < x_1 < \ldots < x_n = 1 \right\}.\end{aligned}$$

We say that the logarithm of $|T'|$ has *bounded variation* if this value $\mathrm{Var}(\log |T'|)$ is finite.

It is easy to see that if $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is C^2 with $|T''|/|T'|$ is bounded then $\mathrm{Var}(\log |T'|)$ is finite.

We now come to the main result of this section which gives sufficient conditions for a homeomorphism to be conjugate to a rotation (and are more readily checked than those of Proposition 6.4).

THEOREM 6.5 (DENJOY'S THEOREM). *If $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a C^1 orientation preserving homeomorphim of the circle with derivative of bounded variation and irrational rotation number $\rho = \rho(T)$ then $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is topologically conjugate to the standard rotation $R_\rho : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.*

PROOF. It suffices to show that T is minimal, then the result follows by applying Proposition 6.4. The proof of minimality will come via two sublemmas.

SUBLEMMA 6.5.1. *If T has irrational rotation number and there are a constant $C > 0$ and a sequence of integers $q_n \to +\infty$ such that the maps $T^{q_n} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ satisfy*

$$|(T^{q_n})'(x)| \cdot |(T^{-q_n})'(x)| \geq C$$

then $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is minimal.

PROOF. If T is not minimal we may choose $x \in \mathbb{R}/\mathbb{Z}$ such that $Y = \mathrm{cl}(\cup_{n\in\mathbb{Z}} T^n x) \neq X$. We can choose a (maximal) interval $I_0 \subset X - Y$; then we claim that $I_n := T^{-n} I_0 \subset X - Y$ are distinct (maximal) intervals. To see this we observe that by maximality I_0 must be of the form $I_0 = (a, b)$ (with $a, b \in Y$). Thus $I_n = (T^{-n}a, T^{-n}b)$ and if $I_n \cap I_m \neq \emptyset$ then again by maximality $I_n = I_m$ and, in particular, $T^{-n}a = T^{-m}a$. But if $n \neq m$ then this means a is a periodic point, which contradicts T having an irrational rotation number (by Lemma 6.1 (ii)).

If $|I_n|$ denotes the length of the interval I_n, $n \in \mathbb{Z}$, then by the disjointness we see that $\sum_{n\in\mathbb{Z}} |I_n| \leq 1$. In particular, $|I_n| \to 0$ as $n \to +\infty$.

However, we see that for all $n \geq 1$,

$$\begin{aligned} |I_{q_n}| + |I_{-q_n}| &= \int_{I_0} \left(|(T^{q_n})'(x)| + |(T^{-q_n})'(x)|\right) dx \\ &\geq 2 \int_{I_0} \left(|(T^{q_n})'(x)||(T^{-q_n})'(x)|\right)^{\frac{1}{2}} dx \\ &\geq 2C^{\frac{1}{2}} |I_0| \end{aligned}$$

(since arithmetic averages are larger than geometric averages). This contradicts $|I_n| \to 0$, and so completes the proof of the sublemma.

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To make use of the assumption of the bounded variation of $\log |T'|$ we need a second sublemma.

SUBLEMMA 6.5.2. *Fix $x \in \mathbb{R}/\mathbb{Z}$ and write $x_n = T^n(x)$, for $n \in \mathbb{Z}$. There exists an increasing sequence $q_n \to +\infty$ of natural numbers such that the intervals*

$$(x_0, x_{q_n}), (x_1, x_{q_n+1}), (x_2, x_{q_n+2}), \ldots, (x_i, x_{q_n+i}), \ldots, (x_{q_n}, x_{2q_n})$$

are all disjoint.

PROOF. By Corollary 6.3.1 we see that the order on $\mathbb{R}/\mathbb{Z}$ of points in the orbit of $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the same as that of the rotation $R_\rho : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. Thus it suffices to prove this sublemma with R_ρ rather than T.

For each $n \geq 1$ we want to choose the sequence q_n to correspond to the successive nearest approaches of $\{T^m x\}$ to x (i.e. $|T^{q_n} x - x| = \delta_n =$

$\inf\{|T^j x - x| : 1 \leq j \leq q_n - 1\}$). Consider a typical interval (x_i, x_{q_n+i}), $0 \leq i \leq q_n$, (the case (x_{q_n+i}, x_i) being similar) and assume for a contradiction that there exists $x_r \in (x_i, x_{q_n+i})$, $0 \leq r \leq 2q_n$. There are two cases.

(a) Firstly, assume that $r < i$. We then know that

$$x_0 = R_\rho^{-r}(x_r) \in R_\rho^{-r}(x_i, x_{q_n+i}) = (x_{(i-r)}, x_{q_n+(i-r)}).$$

In particular, $|x_{i-r} - x_0| < |x_{q_n+(i-r)} - x_{(i-r)}| = |x_{q_n} - x_0| = \delta_n$. But since $q_n > (i-r) > 0$ this contradicts the definition of q_n.

(b) Secondly, assume that $r > i$. We then know that

$$x_{r-i} = R_\rho^{-i}(x_r) \in R_\rho^{-i}(x_i, x_{q_n+i}) = (x_0, x_{q_n})$$

and so we see that $r - i > q_n$ (from the definition of q_n). But then $x_{(r-i)-q_n} = R_\rho^{-q_n}(x_{r-i}) \in R_\rho^{-q_n}(x_0, x_{q_n}) = (x_{-q_n}, x_0)$ and, in particular, $|x_{(r-i)-q_n} - x_0| < |x_{-q_n} - x_0| = |R_\rho^{q_n}(x_{-q_n}) - R_\rho^{q_n}(x_0)| = |x_0 - x_{q_n}| = \delta_n$. Since $0 < (r-i) - q_n < q_n$ this contradicts the definition of q_n.

This completes the proof of the sublemma.

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Since the intervals in Sublemma 6.5.2 are disjoint we have for any $n \geq 1$

$$\begin{aligned}
\mathrm{Var}(\log|T'|) &\geq \sum_{i=0}^{q_n} |\log|T'(x_i)| - \log|T'(x_{q_n+i})|| \\
&\geq \left|\sum_{i=0}^{q_n} \log|T'(x_i)| - \sum_{i=0}^{q_n} \log|T'(x_{q_n+i})|\right| \\
&= \left|\log\left(\frac{\prod_{i=0}^{q_n-1}|T'(T^i x_0)|}{\prod_{i=0}^{q_n-1}|T'(T^i x_{q_n})|}\right)\right| \\
&= \left|\log\left(\frac{|(T^{q_n})'(x_0)|}{|(T^{q_n})'(x_{q_n})|}\right)\right| \\
&= \left|\log|(T^{q_n})'(x_{q_n})(T^{-q_n})'(x_{q_n})|\right|
\end{aligned} \tag{6.4}$$

(where by the chain rule $(T^{-q_n})'(x_{q_n})(T^{q_n})'(x_0) = 1$). Since this holds for arbitrary x the point x_{q_n} can be replaced by an arbitrary point on the circle. If we take the exponential in identity (6.4) then Theorem 6.5 now follows from Sublemma 6.5.1 and 6.4.

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REMARK. We should remark that the assumption that $\log T'$ has bounded variation is necessary. If we relax this assumption then we have that T may be merely semi-conjugate to the rotation R_ρ.

6.3 Comments and references

Some basic results about rotation numbers can be found in [2, pp. 102-108], [5, chapter 12] and [1].

The question of when the conjugating map is differentiable is subtler (cf.[4], [3], [9] and [6], [7], [8] for variants).

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