CHAPTER 2

AN APPLICATION OF RECURRENCE TO ARITHMETIC PROGRESSIONS

In this chapter we shall describe a particularly nice application of the recurrence ideas from chapter 1 to a result in number theory.

2.1 Van der Waerden's theorem

We begin with a simple idea from number theory.

DEFINITION. An *arithmetic progression* is a sequence of integers $\{a + jb\}_{j=0}^{N-1}$ for $a, b \in \mathbb{Z}$ $(b \neq 0)$, $N \geq 1$. We call N the *length* of the arithmetic progression.

EXAMPLES.

(1) The sequence $10, 13, 16, 19, 22$ is an arithmetic progression with $a = 10, b = 3, N = 5$.
(2) The sequence $-4, 0, 4, 8$ is an arithmetic progression with $a = -4, b = 4, N = 4$.

Consider a *partition* of the integers $\mathbb{Z} = B_1 \cup \ldots \cup B_l$ where

(i) $B_i \neq \emptyset$,
(ii) $B_i \cap B_j = \emptyset$ for $i \neq j$.

The main result we want to prove is the following.

THEOREM 2.1 (VAN DER WAERDEN). *Consider a finite partition $\mathbb{Z} = B_1 \cup \ldots \cup B_k$. At least one element B_r in the partition will contain arithmetic progressions of arbitrary length (i.e. $\exists 1 \leq r \leq k$, $\forall N > 0$, $\exists a, b \in \mathbb{Z}$ $(b \neq 0)$ such that $a + jb \in B_r$ for $j = 0, \ldots, N-1$).*

Since an arithmetic progression of length N contains arithmetic progressions of all shorter lengths, this is equivalent to: $\exists N_i \to +\infty$, $\exists a_i, b_i \in \mathbb{Z}$ such that $a_i + jb_i \in B_r$ for $j = 0, \ldots, N_i - 1$.

We give below some simple examples.

EXAMPLES.

(1) If the sets $B_2, \ldots, B_k$, say, in the partition are finite then it is easy to see that B_1 is the element with arithmetic progressions of arbitrary length.

(2) If $\mathbb{Z} = B_1 \cup B_2$ where $B_1 = \{\text{odd numbers}\}$ and $B_2 = \{\text{even numbers}\}$ then both contain arithmetic progressions of arbitrary length.
(3) If $B_1 = \{\text{prime numbers}\}$ and $B_2 = \{\text{non-prime numbers}\}$ then B_2 contains arithmetic progressions of arbitrary length. However, it is an unsolved problem as to whether B_1 contains arithmetic progressions of arbitrary length.

Historical Note. This result was originally conjectured by Baudet and proved by Van der Waerden in 1927 [6, 7]. The theorem gained a wider audience when it was included in Khintchine's famous book *Three pearls in number theory* [4]. The dynamical proof we give is due to Furstenberg and Weiss [3](from 1978).

2.2 A dynamical proof

The key to proving Van der Waerden's theorem is the following generalization of Birkhoff's theorem.

Theorem 2.2. *Let $T_1, \ldots, T_N : X \to X$ be homeomorphisms of a compact metric space such that $T_i T_j = T_j T_i$ for $1 \le i, j \le N$. There exist $x \in X$ and $n_j \to +\infty$ such that $d(T_i^{n_j} x, x) \to 0$ for each $i = 1, \ldots, N$.*

We shall first prove Theorem 2.1 assuming Theorem 2.2 and then return to the proof of Theorem 2.2.

Proof of Theorem 2.1 (assuming Theorem 2.2). We want to begin by associating to the partition $\mathbb{Z} = B_1 \cup \ldots \cup B_k$ a suitable homeomorphism $T : X \to X$ (and then we set $T_j = T^j$, $j = 1, \ldots, N$).

Let $\Omega = \prod_{n \in \mathbb{Z}} \{1, \ldots, k\}$ and then we can associate to the partition $\mathbb{Z} = B_1 \cup \ldots \cup B_k$ a sequence $z = (z_n)_{n \in \mathbb{Z}} \in \Omega$ by $z_n = i$ if and only if $n \in B_i$.

Let $\sigma : \Omega \to \Omega$ be the shift introduced in Example 3 of section 1.1 (i.e. $(\sigma x)_n = x_{n+1}$, $n \in \mathbb{Z}$). Consider the orbit $\{\sigma^n z : n \in \mathbb{Z}\}$ and its closure $X = \mathrm{cl}\,(\cup_{n \in \mathbb{Z}} \sigma^n z)$. Finally, we define $T_i := T^i = \sigma \circ \ldots \circ \sigma$ (T composed with itself i times).

By Theorem 2.2 (with $\epsilon = \frac{1}{4}$) we can find $x \in X$ and $b \ge 1$ with

$$d(T_1^b x, x) < \frac{1}{4}, d(T_2^b x, x) < \frac{1}{4}, \ldots, d(T_N^b x, x) < \frac{1}{4}.$$

Since $X = \mathrm{cl}\,(\cup_{n \in \mathbb{Z}} \sigma^n z)$ we can choose $a \in \mathbb{Z}$ such that

$$d(x, T^a z) < \frac{1}{4}, d(T_1^b x, T^a T_1^b z) < \frac{1}{4}, \ldots, d(T_N^b x, T^a T_N^b z) < \frac{1}{4}.$$

Thus, for each $i = 1, \ldots, N$ we have that

$$d(T^a T_i^b x, T^a z) \le d(T^a T_i^b x, T_i^b x) + d(T_i^b x, x) + d(x, T^a z) < \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Since $d(x,y) = \left(\frac{1}{2}\right)^{N(x,y)}$ (where $N(x,y) = \min\{|N| \geq 0 : x_N \neq y_N, \text{ or } x_{-N} \neq y_{-N}\}$) we see that $(T^b T_i^a x)_0 = x_{b+ia} = z_a \in \{1, \ldots, k\}$ for $i = 1, \ldots, N$. This means that $b + ia \in B_{z_a}$, for $i = 1, \ldots, N$, and completes the proof of Theorem 2.1.

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All that remains is to prove Theorem 2.2. This is a fairly detailed proof and to help clarify matters we shall divide it into sublemmas.

Proof of Theorem 2.2. We shall use a proof by induction.

Case $N = 1$. For $N = 1$ the multiple Birkhoff recurrence theorem reduces to the (usual) Birkhoff recurrence theorem (Corollary 1.8.1).

Inductive step. Assume that the result is known for $N - 1$ commuting homeomorphisms. We need to show that it holds for N commuting homeomorphisms.

Simplifying fact. We can assume that X is the *smallest* closed set invariant under each of $T_1, \ldots, T_N$. If this is not the case we can restrict to such a set (using Zorn's lemma as in section 1.6).

In order to establish the Birkhoff multiple recurrence theorem for these N commuting homeomorphisms, the following simple alternative formulation of this result is useful.

Alternative formulation. Let $\mathcal{X}_N = X \times \ldots \times X$ be the N-fold cartesian product of X and let $\mathcal{D}_N = \{(x, \ldots, x) \in \mathcal{X}_N\}$ be the diagonal of the space. Let $S : \mathcal{X}_N \to \mathcal{X}_N$ be given by $S(x_1, \ldots, x_N) = (T_1 x_1, \ldots, T_N x_N)$. Then the following are equivalent:

$(i)_N$ the Birkhoff multiple recurrence holds for $T_1, \ldots, T_N$;
$(ii)_N$ $\exists \underline{z} = (z, \ldots, z) \in \mathcal{D}_N$ such that $d_{\mathcal{X}_N}(S^{n_i} \underline{z}, \underline{z}) \to 0$ as $n_i \to +\infty$

(where $d_{\mathcal{X}_N}(\underline{z}, \underline{w}) = \sup_{1 \leq i \leq N} d(z_i, x_i)$).

We can apply the inductive hypothesis to the $(N-1)$ commuting homeomorphisms $T_1 T_N^{-1}, \ldots, T_{N-1} T_N^{-1}$ and using the equivalence of $(i)_{N-1}$ and $(ii)_{N-1}$ above we have that for the map $R := T_1 T_N^{-1} \times \ldots \times T_{N-1} T_N^{-1} : \mathcal{X}_{N-1} \to \mathcal{X}_{N-1}$ defined by

$$R : (x_1, \ldots, x_{N-1}) \mapsto (T_1 T_N^{-1} x_1, \ldots, T_{N-1} T_N^{-1} x_{N-1})$$

there exists $\underline{z} = (z, \ldots, z) \in \mathcal{D}_{N-1} \subset \mathcal{X}_{N-1}$ with $d_{\mathcal{X}_{N-1}}(R^{n_i} \underline{z}, \underline{z}) \to 0$ as $n_i \to +\infty$. In particular, $d_{\mathcal{X}_N}(S^{n_i} \underline{z}', z) \to 0$ as $n_i \to +\infty$ where $\underline{z} = (z, \ldots, z), \underline{z}' = (T_N^{-n_i} z, \ldots, T_N^{-n_i} z) \in \mathcal{D}_N$.

Thus we have proved the following result.

SUBLEMMA 2.2.1. *$\forall \epsilon > 0$, $\exists z, z' \in \mathcal{D}_N$, $\exists n \geq 1$ such that $d_{\mathcal{X}_N}(S^n z, z') < \epsilon$.*

Unfortunately, this is not quite in the form of $(ii)_N$ we need for the inductive step. (For example, we would like to take $z = z'$.) To get a stronger result, we break the argument up into steps represented by the following sublemmas.

SUBLEMMA 2.2.2. *$\forall \epsilon > 0, \forall x \in \mathcal{D}_N, \exists y \in \mathcal{D}_N$ and $\exists n \geq 1$ such that $d(S^n y, x) < \epsilon$.*

(This changes one of the quantifiers $\exists$ to $\forall$.)

SUBLEMMA 2.2.3. *$\forall \epsilon > 0, \exists z \in \mathcal{D}_N$ and $n \geq 1$ such that $d(S^n z, z) < \epsilon$*

(This is almost the Birkhoff multiple recurrence theorem, except that z might still depend on the choice of $\epsilon > 0$.)

We will now complete the proof of the Birkhoff multiple recurrence theorem assuming Sublemma 2.2.3. (We shall then return to the proofs "Sublemma 2.2.1 $\implies$ Sublemma 2.2.2" and "Sublemma 2.2.2 $\implies$ Sublemma 2.2.3" in the next section.)

Consider the function $F : \mathcal{D}_N \to \mathbb{R}^+ = [0, +\infty)$ defined by $F(x) = \inf_{n \geq 1} d(S^n x, x)$. It is easy to see that to complete the proof of Theorem 2.2 we need only show there exists a point $x_0 \in \mathcal{D}_N$ with $F(x_0) = 0$. To show this fact, the following properties of F are needed.

SUBLEMMA 2.2.4.

(i) *$F : \mathcal{D}_N \to \mathbb{R}^+$ is upper semi-continuous (i.e. $\forall x \in \mathcal{D}_N, \forall \epsilon > 0, \exists \delta > 0$ such that $d(x, y) < \delta \implies F(y) \leq F(x) + \epsilon$).*

(ii) *$\exists x_0 \in \mathcal{D}_N$ such that $F : \mathcal{D}_N \to \mathbb{R}^+$ is continuous at x_0.*

PROOF.

(i) This is an easy exercise from the definition of F.

(ii) For $\epsilon > 0$ we can define $A_\epsilon = \{x \in \mathcal{D}_N : \forall \eta > 0, \exists y$ such that $d(y, x) < \eta$ and $F(y) \leq F(x) - \epsilon\}$ (i.e. $\exists$ points y arbitrarily close to x with $F(y) \leq F(x) - \epsilon$). Notice that

(a) A_ϵ is closed,

(b) A_ϵ has empty interior.

(To see part (b) observe that if $\mathrm{int}(A_\epsilon) \neq \emptyset$ we could choose a sequence of pairs $x, x_1 \in \mathrm{int}(A_\epsilon)$ with $F(x_1) \leq F(x) - \epsilon$, $x_1, x_2 \in \mathrm{int}(A_\epsilon)$ with $F(x_2) \leq F(x_1) - \epsilon$, etc. Together these inequalities give $F(x_n) \leq F(x) - n\epsilon < 0$ for n arbitrarily large. But this contradicts $F \geq 0$).

The set of points at which F is continuous is

$$\{x \in \mathcal{D}_N\colon x \notin A_\epsilon, \epsilon > 0\} = \cap_{n=1}^{\infty} \left(\mathcal{D}_N - A_{\frac{1}{n}}\right).$$

Since this is a countable intersection of open dense sets, it is still dense (by Baire's theorem). Thus there exists at least one point of continuity for $F : \mathcal{D}_N \to \mathbb{R}^+$ (in fact, infinitely many). This completes the proof of Sublemma 2.2.4.

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Let x_0 be such a point of continuity.

Assume for a contradiction that $F(x_0) > 0$. We can then choose $\delta > 0$ and an open neighbourhood $U \ni x_0$ such that $F(x) > \delta > 0$ for $x \in U$. However, we also know that for the diagonal actions $T_i : (x_1, \dots, x_N) \mapsto (T_i x_1, \dots, T_i x_N)$

$$\mathcal{D}_N \subset \cup_{j=1}^M \left(T_1^{n_{1j}} \circ \dots \circ T_N^{n_{Nj}}\right)^{-1} U$$

(since by the simplifying assumption X is the smallest closed set invariant under $T_1, \dots, T_N$ and so we may apply Lemma 1.9 from Chapter 1).

By (uniform) continuity of the family $\{T_1^{n_{1j}} \circ \dots \circ T_N^{n_{Nj}}\}_{j=1}^M$ there exists $\eta > 0$ such that

$$d(x,y) < \eta \implies d(T_1^{n_{1j}} \circ \dots \circ T^{n_{Nj}} x, T_1^{n_{1j}} \circ \dots \circ T_N^{n_{Nj}} y) < \delta \tag{2.1}$$

(for $1 \leq j \leq M$). Observe that for $y \in \left(T_1^{n_{1j}} \circ \dots \circ T_N^{n_{Nj}}\right)^{-1} U$ ($j = 1, \dots, M$) we have that $F(y) \geq \eta$. If this were not the case then there would exist $n \geq 1$ with $d(y, S^n y) < \eta$, from the definition of F. This then implies that $d(T_1^{n_{1j}} \circ \dots \circ T_N^{n_{Nj}} y, T_1^{n_{1j}} \circ \dots \circ T_N^{n_{Nj}} S^n y) < \delta$ by (2.1). Choosing $x := T_1^{n_{1j}} \circ \dots \circ T_N^{n_{Nj}} y \in U$ gives $F(x) = \inf_{n \geq 1} d(x, S^n x) < \delta$ which contradicts our hypothesis.

Finally we see that by (2.1) we have $F(y) \geq \eta$ for all $y \in \mathcal{D}_N$. However, this contradicts Sublemma 2.2.3 and we conclude that $F(x_0) = 0$.

The proof of Theorem 2.3 is finished (given the proofs of Sublemma 2.2.2 and Sublemma 2.2.3).

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2.3. The proofs of Sublemma 2.2.2 and Sublemma 2.2.3

We now supply the missing proofs of Sublemma 2.2.2 and Sublemma 2.2.3.

Proof of Sublemma 2.2.2 (assuming Sublemma 2.2.1). Consider the N commuting maps $\hat{T}_1, \hat{T}_2, \dots, \hat{T}_N : \mathcal{D}_N \to \mathcal{D}_N$ defined by

$$\begin{cases} \hat{T}_1 = T_1 \times \dots \times T_1 : \mathcal{D}_N \to \mathcal{D}_N, \\ \hat{T}_2 = T_2 \times \dots \times T_2 : \mathcal{D}_N \to \mathcal{D}_N, \\ \qquad \dots \\ \hat{T}_N = T_N \times \dots \times T_N : \mathcal{D}_N \to \mathcal{D}_N. \end{cases}$$

We want to apply Lemma 1.9 to these commuting maps with the choice of open set $U = \{w \in \mathcal{D}_N : d_{\mathcal{D}_N}(x, w) < \frac{\epsilon}{2}\}$. This allows us to conclude that there exist $n_{1j}, \dots, n_{Nj}$ $(j = 1, \dots, M)$ such that

$$\mathcal{D}_N = \cup_{j=1}^{N} \hat{T}^{-n_{1j}} \dots \hat{T}^{-n_{Nj}} U.$$

Thus for any $z \in \mathcal{D}_N$ we have some $1 \leq j \leq M$ such that

$$d_{\mathcal{D}_N}(\hat{T}^{n_{1j}} \dots \hat{T}^{n_{Nj}} z, x) < \frac{\epsilon}{2}. \tag{2.2}$$

Next we can use (uniform) continuity of $\hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}}$ to say that there exsits $\delta > 0$ such that whenever $d_{\mathcal{D}_N}(z, z') < \delta$ for $z, z' \in \mathcal{D}_N$ then we have that

$$d_{\mathcal{X}_N}(\hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}} z, \hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}} z') < \frac{\epsilon}{4}. \tag{2.3}$$

By Sublemma 2.2.1 $\exists z, z' \in \mathcal{D}_N$ and $\exists n \geq 1$ such that $d_{\mathcal{X}_N}(S^n z, z') < \delta$. Therefore by inequality (2.3) we have that

$$d_{\mathcal{X}_N}\left(S^n\left(\hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}} z\right), \hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}} z'\right) < \frac{\epsilon}{4}. \tag{2.4}$$

Writing $y = \hat{T}^{n_{1j}} \dots \hat{T}^{n_{Nj}} z$ and comparing (2.2), (2.3) and (2.4) gives that

$$\begin{aligned} d_{\mathcal{X}_N}(S^n y, x) &\leq d_{\mathcal{X}_N}(S^n y, \hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}} z') + d_{\mathcal{X}_N}(\hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}} z', x) \\ &\quad + d_{\mathcal{X}_N}(\hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}} z, \hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}} z') \\ &\quad + d_{\mathcal{X}_N}(\hat{T}^{n_{1j}} \circ \dots \circ \hat{T}^{n_{Nj}} z, x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

This completes the proof of Sublemma 2.2.2.

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Proof of Sublemma 2.2.3 (assuming Sublemma 2.2.2). Fix $z_0 \in \mathcal{D}_N$ and let $\epsilon_1 = \frac{\epsilon}{2}$. By Sublemma 2.2.2 we can choose $n_1 \geq 1$ and $z_1 \in \mathcal{D}_N$ with $d(T^{n_1} z_1, z_0) < \epsilon_1$.

By continuity of T^{n_1} we can find $\epsilon_1 > \epsilon_2 > 0$ such that $d(z, z_1) < \epsilon_2$ implies that $d(T^{n_1} z, z_0) < \epsilon_1$.

We can now continue inductively (for $k \geq 2$):

(a) By Sublemma 2.2 we can choose $n_k \geq 1$ and $z_k \in \mathcal{D}_N$ with $d(T_k^n z_k, z_{k-1}) < \epsilon_k$.
(b) By continuity of T^{n_k} we can find $\epsilon_k > \epsilon_{k+1} > 0$ such that $d(z, z_k) < \epsilon_{k+1}$ implies that $d(T^{n_k} z, z_{k-1}) < \epsilon_k$.

This results in sequences

$$\begin{array}{l} z_0, z_1, z_2, \ldots \in \mathcal{D}_N, \\ n_0, n_1, n_2 \ldots \in \mathbb{N}, \\ \epsilon_0 > \epsilon_1 > \epsilon_2 > \ldots \end{array} \quad \text{such that} \left\{ \begin{array}{l} d(T^{n_k} z_k, z_{k-1}) < \epsilon_k, k \leq 1, \\ d(z, z_i) < \epsilon_{k+1} \implies d(T^{n_k} z, z_{k-1}) < \epsilon_k. \end{array} \right.$$

In particular we get that whenever $j < i$ then

$$d(T^{n_i + n_{i-1} + \ldots + n_{j+2} + n_{j+1}} z_i, z_j) < \epsilon_{i+1} \leq \frac{\epsilon}{2}. \tag{2.5}$$

By compactness of $\mathcal{D}_N$ we can find $d(z_i, z_j) < \frac{\epsilon}{2}$ for some $j < i$.

By the triangle inequality we have that for $N = n_i + n_{i-1} + \ldots + n_{j+1}$

$$d(T^N z_i, z_i) \leq d(T^N z_i, z_j) + d(z_j, z_i) < \epsilon.$$

Thus the choice $z = z_i$ completes the proof of Sublemma 2.2.3.

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2.4 Comments and references

A treatment of Van der Waerden's theorem (and many other related applications of dynamics to number theory) can be found in [1]. The proof originally appeared in the article [3] and the survey [2]. An account also appears in [5].

Sublemma 2.2.2 was originally proved by Bowen.

Finally, there is a stronger version of this result due to Szemeredi. In chapter 16 we shall present Furstenburg's proof of this using ergodic theory.

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