CHAPTER 15

INVARIANT MEASURES FOR COMMUTING TRANSFORMATIONS

In this chapter we describe an important conjecture of Furstenberg and related work of Rudolph.

15.1 Furstenberg's conjecture and Rudolph's theorem

Consider the transformations

(i) $S : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by $S(x) = 2x \pmod 1$, and
(ii) $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by $T(x) = 3x \pmod 1$.

(For a mnemonic aid: S stands for "second" and T for "third".) It is easy to see that these transformations commute, i.e. $ST = TS$).

Recall that the S-invariant probability measures form a convex weak-star compact set $\mathcal{M}_S$ (and similarly, the T-invariant probability measures form a convex weak-star compact set $\mathcal{M}_T$).

We want to describe the probability measures which are both T-invariant and S-invariant (i.e. the intersection $\mathcal{M}_S \cap \mathcal{M}_T$). We need only consider the (S,T)-ergodic measures μ in $\mathcal{M}_S \cap \mathcal{M}_T$ (i.e. those probability measures invariant under both S and T for which the only Borel sets B with $T^{-n}S^{-m}B = B$ $\forall n, m \geq 0$ have either $\mu(B) = 0$ or 1, since these are the extremal measures in $\mathcal{M}_S \cap \mathcal{M}_T$).

FURSTENBERG'S CONJECTURE. *The only (S,T)-ergodic measures are the Haar-Lebesgue measure and measures supported on a finite set.*

Notice that the Haar-Lebesgue measure ν has entropies $\log 2$ and $\log 3$, respectively, for the transformations S and T, and any finitely supported measure always has zero entropy with respect to either S or T. The following partial solution is due to D.J. Rudolph.

THEOREM 15.1 (RUDOLPH). *The only (S,T)-ergodic measure μ which has non-zero entropy (w.r.t. either S or T) is the Haar-Lebesgue measure.*

15.2 The proof of Rudolph's theorem

We begin with a few comments.

(1) Haar-Lebesgue measure ν on the unit circle is characterized as the only probability measure invariant under all rotations on the circle.

Moreover, it is the only measure invariant under all rotations $x \mapsto x + a \pmod 1$, where a is any rational number $a = \frac{j}{2^k}$.

(2) For $n \geq 1$ and $f \in L^2(X, \mathcal{B}, \mu)$ we can write that

$$E(f|T^{-n}\mathcal{B})(x) = \sum_{y \in T^{-n}T^n x} \frac{f(y)}{(T^n)'(y)}$$

where $T'(x) = \frac{d\mu T}{d\mu}$ (and similarly for S).

(3) We can write $(S^n)'(x) = S'(S^{n-1}x) \dots S'(x)$. If we knew that S' is constant (almost everywhere) then by the martingale theorem we would have that

$$\begin{aligned}
\int f(x+a) d\mu &= \lim_{n \to +\infty} \int E(f(x+a)|S^{-n}\mathcal{B}) \\
&= \lim_{n \to +\infty} \int E(f(x)|S^{-n}\mathcal{B}) \\
&= \int f(x) d\mu
\end{aligned}$$

and thus we know that ν is the Haar-Lebesgue measure.

(4) Since $ST = TS$ we have that

$$T'(Sx) \cdot S'(x) = (TS)'(x) = (ST)'(x) = S'(Tx) \cdot S'(x).$$

In particular, we can write $\frac{S'(Tx)}{T'(Sx)} = \frac{S'(x)}{T'(x)}$.

We begin with the following simple (but fundamental) Sub-lemma.

SUB-LEMMA 15.1.1. *$S'(Tx) = S'(x)$ for almost all x.*

PROOF. We begin by claiming that $E(S'|T^{-1}\mathcal{B})(x) = S'(Tx)$. To see this we observe that

$$\begin{aligned}
E(S'|T^{-1}\mathcal{B})(x) &= \sum_{y:Ty=Tx} \frac{S'(y)}{T'(y)} \\
&= \sum_{y:Ty=Tx} \frac{S'(Ty)}{T'(Sy)} \text{ (by (4) above)} \\
&= S'(Tx) \left(\sum_{y:Ty=Tx} \frac{1}{T'(Sy)} \right) \\
&= S'(Tx)
\end{aligned}$$

where we have used that there is a bijection between $\{y : Ty = Tx\}$ and $\{w : Tw = T(Sx)\}$ to write that

$$\sum_{y:Ty=Tx} \frac{1}{T'(Sy)} = \sum_{w:Tw=T(Sx)} \frac{1}{T'(w)} = 1.$$

This proves the claim. To complete the proof of the Sub-lemma we need only show that $E(S'|T^{-1}\mathcal{B})(x) = S'(x)$. However, since $E(.|T^{-1}\mathcal{B})(x) : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ is a positive operator which is a contraction and

$$||E(S'|T^{-1}\mathcal{B})||_2 = ||S'T|_2| = ||S'||_2,$$

we indeed see that $S'T = E(S'|T^{-1}\mathcal{B})(x) = S'(x)$.

■

If we knew that T was ergodic we could now deduce S' is constant. Unfortunately, we don't know this and a little more work is required.

DEFINITION. We let $\mathcal{A}_1 \subset \mathcal{B}$ denote the *smallest* sub-sigma-algebra for which $S'(x)$ is measurable.

In the course of the proof we shall establish that $\mathcal{A}_1 \subset S^{-1}\mathcal{B}$ (which, by the definition of $\mathcal{A}_1$, will imply that $S'(x)$ is constant).

Similarly, we can introduce the sub-sigma-algebras $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset \mathcal{A}_n \subset \ldots \subset \mathcal{B}$ where $\mathcal{A}_n$ is the smallest sub sigma-algebra for which all of the functions $S'(x)$, $(S^2)'(x)$, ... ,$(S^n)'(x)$ are measurable.

SUB-LEMMA 15.1.2. *For each $n \geq 1$ we have that*

(a) $S^{-1}\mathcal{A}_n \subset \mathcal{A}_{n+1}$,
(b) $T^{-1}\mathcal{A}_n = \mathcal{A}_n$.

PROOF.

(a) If we write $(S^{n+1})'(x) = (S^n)'(Sx) \cdot S'(x)$ then, since by hypothesis S^n and S' are $\mathcal{A}_n$-measurable, the right hand side is measurable with respect to $S^{-1}\mathcal{A}_n$.
(b) Since $S'(Tx) = S'(x)$ we also see that

$$\begin{aligned}(S^n)'(Tx) &= S'(Tx) \cdot S'(STx) \ldots S'(S^{n-1}Tx)\\ &= S'(x) \cdot S'(Sx) \ldots S'(S^{n-1}x)\\ &= (S^n)'(x).\end{aligned}$$

But the right hand side is $\mathcal{A}_n$-measurable by hypothesis.

■

DEFINITION. We write $\mathcal{A} = \vee_{n=1}^{\infty} \mathcal{A}_n$. The above lemma guarantees that $T^{-1}\mathcal{A} = \mathcal{A}$ and $S^{-1}\mathcal{A} \subset \mathcal{A}$.

We now move on to entropy considerations. Let $\gamma = \{[0, \frac{1}{6}], [\frac{1}{6}, \frac{2}{6}], .., [\frac{5}{6}, 1]\}$ denote the partition into intervals of length one sixth.

SUB-LEMMA 15.1.3. *There exists a sequence $s_n \to +\infty$ such that the partitions*

$$\vee_{i=0}^{s_n} S^{-i}\gamma = \left\{ \left[0, \frac{1}{6 \cdot 2^{s_n}}\right], \left[\frac{1}{6 \cdot 2^{s_n}}, \frac{2}{6 \cdot 2^{s_n}}\right], \dots, \left[\frac{6 \cdot 2^{s_n} - 1}{6 \cdot 2^{s_n}}, 1\right] \right\} \text{ and}$$

$$\vee_{i=0}^{n} T^{-i}\gamma = \left\{ \left[0, \frac{1}{6 \cdot 3^{n}}\right], \left[\frac{1}{6 \cdot 3^{n}}, \frac{2}{6 \cdot 3^{n}}\right], \dots, \left[\frac{6 \cdot 3^{n} - 1}{6 \cdot 3^{n}}, 1\right] \right\}$$

have the property that every element of either partition is contained in at most four elements of the other partition.

PROOF. For each $n \geq 1$ we choose the values $s_n \geq 1$ such that $3^{n-1} \leq 2^{s_n} \leq 3^n$. The lengths of the intervals for each partition are $\frac{1}{6 \cdot 2^{s_n}}$ and $\frac{1}{6 \cdot 3^n}$ and thus their ratios are bounded above and below by 3 and $\frac{1}{3}$, respectively. This is enough to complete the proof.

■

In what follows we shall make frequent use of the basic identity for entropy: $H(\alpha \vee \beta | \mathcal{C}) = H(\alpha | \mathcal{C} \vee \beta) + H(\beta | \mathcal{C})$.

Recall that the *entropies* of the transformations are given by

$$h(T) := \lim_{n \to +\infty} \frac{1}{n} H(\vee_{i=0}^{n-1} T^{-i}\gamma)$$

and

$$h(S) := \lim_{k \to +\infty} \frac{1}{k} H(\vee_{i=0}^{k-1} S^{-i}\gamma).$$

The following sub-lemma shows similar limits involving the sigma-algebra $\mathcal{A}$.

SUB-LEMMA 15.1.4. *The following limits exist and are equal to the entropies:*

$$h(T) = \lim_{n \to +\infty} \frac{1}{n} H(\vee_{i=0}^{n-1} T^{-i}\gamma | \mathcal{A})$$

and

$$h(S) = \lim_{n \to +\infty} \frac{1}{s_n} H(\vee_{i=0}^{s_n - 1} S^{-i}\gamma | \mathcal{A}).$$

PROOF. We begin with an argument which is borrowed from the standard entropy identities. We see that for any $n, m \geq 0$ we have that

$$
\begin{aligned}
&H(\vee_{i=0}^{n+m-1} T^{-i}\gamma|\mathcal{A}) \\
&= H(\vee_{i=0}^{n-1} T^{-i}\gamma|\mathcal{A}) + H(\vee_{i=n}^{n+m-1} T^{-i}\gamma|\mathcal{A} \vee (\vee_{i=0}^{n-1} T^{-i}\gamma)) \\
&\leq H(\vee_{i=0}^{n-1} T^{-i}\gamma|\mathcal{A}) + H(\vee_{i=n}^{n+m-1} T^{-i}\gamma|\mathcal{A}) \\
&= H(\vee_{i=0}^{n-1} T^{-i}\gamma|\mathcal{A}) + H(\vee_{i=0}^{m-1} T^{-i}\gamma|\mathcal{A})
\end{aligned}
$$

(where for the last equality we use that $T^{-1}\mathcal{A} = \mathcal{A}$). Thus by subadditivity the limit $h(T|\mathcal{A}) := \lim_{n\to+\infty} \frac{1}{n} H(\vee_{i=0}^{n-1} T^{-i}\gamma|\mathcal{A})$ exists. By the basic equalities for entropy we see that

$$
\begin{aligned}
&H(\vee_{i=0}^{s_n-1} S^{-i}\gamma|\mathcal{A}) - H(\vee_{i=0}^{n-1} T^{-i}\gamma|\mathcal{A}) \\
&= H\left((\vee_{i=0}^{s_n} S^{-i}\gamma) \vee (\vee_{i=0}^{n-1} T^{-i}\gamma)\,|\mathcal{A}\right) + H\left(\vee_{i=0}^{s_n} S^{-i}\gamma|\,(\vee_{i=0}^{n-1} T^{-i}\gamma) \vee \mathcal{A}\right) \\
&- H\left((\vee_{i=0}^{s_n} S^{-i}\gamma) \vee (\vee_{i=0}^{n-1} T^{-i}\gamma)\,|\mathcal{A}\right) - H\left(\vee_{i=0}^{n-1} T^{-i}\gamma|\,(\vee_{i=0}^{s_n} S^{-i}\gamma) \vee \mathcal{A}\right)
\end{aligned}
$$

and so we can identify the limit as

$$
\begin{aligned}
h(T|\mathcal{A}) &= \lim_{n\to+\infty} \frac{1}{n} H(\vee_{i=0}^{n-1} T^{-i}\gamma|\mathcal{A}) \\
&= \lim_{n\to+\infty} \frac{1}{n} \left(H\left(\vee_{i=0}^{n-2} T^{-i}\gamma|\mathcal{A}\right) + H\left(T^{-(n-1)}\gamma|\mathcal{A} \vee \left(\vee_{i=0}^{n-2} T^{-i}\gamma\right)\right)\right) \\
&= h(T)
\end{aligned}
$$

since

$$
h(T) = \lim_{n\to+\infty} \frac{1}{n} \left(H(\vee_{i=0}^{n-2} T^{-i}\gamma)\right)
$$

and

$$
H\left(T^{-(n-1)}\gamma|\mathcal{A} \vee \left(\vee_{i=0}^{n-2} T^{-i}\gamma\right)\right) \leq H\left(T^{-(n-1)}\gamma|\mathcal{A}\right) = H(\gamma|\mathcal{A}) < +\infty.
$$

By sublemma 15.1.3 we have that the final expression above is bounded (independently of n) and thus we have that the following limit exists

$$
h(S|\mathcal{A}) := \lim_{n\to+\infty} \frac{1}{n} H(\vee_{i=0}^{s_n-1} T^{-i}\gamma|\mathcal{A}).
$$

Moreover, this argument gives that $h(T|\mathcal{A}) = \frac{\log 3}{\log 2} h(S|\mathcal{A})$.

Observe that if we replace $\mathcal{A}$ by the trivial sigma-algebra then the same argument gives that $h(T) = \frac{\log 3}{\log 2} h(S)$. Comparing these identities we see that $h(S) = h(S|\mathcal{A})$.

■

We now apply Sub-lemma 15.1.4 to show that $\mathcal{A} \subset \vee_{i=1}^{\infty} S^{-i}\gamma$, which is essentially the end of the proof.

SUB-LEMMA 15.1.5. $H(\mathcal{A}| \vee_{i=1}^{\infty} S^{-1}\mathcal{B}) = 0$.

PROOF. By the basic equality for entropy we have that

$$\begin{aligned} H(\mathcal{A}| \vee_{i=1}^{\infty} S^{-i}\gamma) &= H(\gamma \vee \mathcal{A}| \vee_{i=1}^{\infty} S^{-i}\beta) - H(\gamma| \vee_{i=1}^{\infty} S^{-i}\gamma \vee \mathcal{A}) \\ &= H(\gamma| \vee_{i=1}^{\infty} S^{-1}\beta) - H(\gamma| \vee_{i=1}^{\infty} S^{-i}\gamma \vee \mathcal{A}) \\ &= h(S) - H(\gamma| \vee_{i=1}^{\infty} S^{-i}\gamma \vee \mathcal{A}) \end{aligned} \tag{15.1}$$

(where we have used that $H(\gamma \vee \mathcal{A}| \vee_{i=1}^{\infty} S^{-i}\beta) = H(\gamma| \vee_{i=1}^{\infty} S^{-i}\beta) = h(S)$). We next observe that

$$\begin{aligned} & H(\vee_{i=0}^{n-1} S^{-i}\gamma|\mathcal{A}) \\ &= H(\gamma| \vee_{i=1}^{n-1} S^{-i}\gamma \vee \mathcal{A}) + H(\vee_{i=1}^{n-1} S^{-i}\gamma|\mathcal{A}) \\ &\leq H(\gamma| \vee_{i=1}^{n-1} S^{-i}\gamma \vee \mathcal{A}) + H(\vee_{i=0}^{n-2} S^{-i}\gamma|\mathcal{A}) \\ &\dots \\ &\leq H(\gamma| \vee_{i=1}^{n-1} S^{-i}\gamma \vee \mathcal{A}) + H(\gamma| \vee_{i=1}^{n-2} S^{-i}\gamma \vee \mathcal{A}) + \ldots + H(\gamma|\mathcal{A}) \end{aligned}$$

(This argument is a modification of the standard entropy proof that $h(S) = H(\gamma| \vee_{i=1}^{\infty} S^{-i}\gamma)$.) Thus from the definition of $h(S|\mathcal{A})$ we have that

$$\begin{aligned} h(S|\mathcal{A}) &:= \lim_{n\to+\infty} \frac{1}{n} H(\vee_{i=0}^{n-1} S^{-i}\gamma|\mathcal{A}) \\ &= \lim_{n\to+\infty} \frac{1}{n} H(\gamma| \vee_{i=1}^{n-1} S^{-i}\gamma \vee \mathcal{A}) + \lim_{n\to+\infty} \frac{1}{n} H(\vee_{i=1}^{n-1} S^{-i}\gamma|\mathcal{A}) \\ &= H(\gamma| \vee_{i=1}^{\infty} S^{-i}\gamma). \end{aligned} \tag{15.2}$$

Comparing (15.1) and (15.2) we see that

$$0 \leq H(\gamma| \vee_{i=1}^{\infty} S^{-i}\gamma) \leq h(T) - h(T|\mathcal{A}) = 0.$$

■

To finish off the proof of Theorem 15.1 we need only recall that $H(\mathcal{A}| \vee_{i=1}^{\infty} S^{-i}\gamma) = 0$ implies that $\mathcal{A} \subset \vee_{i=1}^{\infty} S^{-i}\gamma$.

Repeating the argument with S replaced by S^k for $k = 1, 2, \ldots$ we see that $\mathcal{A} \subset \cap_{n=0}^{\infty} S^{-n}\mathcal{B}$. In particular, this shows that $(S^n)'(y)$ is constant for $y \in \{w | S^n x = S^n w\}$ (almost everywhere).

We observe that since $h(S) > 0$ (equivalently $h(T) > 0$), there must be a set of positive measure on which S has two pre-images (otherwise S would be invertible almost everywhere and then have entropy zero). Moreover we claim that the set with two S pre-images is invariant under S and T. By *ergodicity* of (S, T) we see that almost all points have two pre-images.

This suffices to apply the argument in comment (3).

■

15.3 Comments and references

The original proof of Rudolph had a symbolic formulation [2]. The proof we give here is a version due to Parry [1].

References

1. W. Parry, *Squaring and cubing the circle*, Ergodic Theory, Proceedings of the Warwick Symposium on $\mathbb{Z}^d$-actions (M. Pollicott and K. Schmidt, ed.), C.U.P., Cambridge, 1996, pp. 177-183.
2. D. Rudolph, $\times 2$ *and* $\times 3$ *invariant measures and entropy*, Ergod. Th. and Dynam. Sys. **10** (1990), 395-406.