CHAPTER 13

FIXED POINTS FOR HOMEOMORPHISMS OF THE ANNULUS

13.1 Fixed points for the annulus

Let $A = \mathbb{R}/\mathbb{Z} \times [0,1]$ be a closed annulus. Assume that $T : A \to A$ is a homeomorphism that preserves the two boundary circles (i.e. $T(\mathbb{R}/\mathbb{Z}\times\{0\}) = \mathbb{R}/\mathbb{Z} \times \{0\}$ and $T(\mathbb{R}/\mathbb{Z} \times \{1\}) = \mathbb{R}/\mathbb{Z} \times \{1\}$).

DEFINITION. We say that $T : A \to A$ is *area preserving* if the Haar-Lebesgue measure λ is T-invariant.

REMARK. We say that $T : A \to A$ is *conservative* if there are no wandering sets of positive measure. This condition will suffice for most of the results of this section.

EXAMPLE. For any pair of values $0 < \alpha, \beta < 1$ consider the map $T : A \to A$ given by $T(x,y) = (x+\alpha y+\beta(1-y), y)$. To see that this is area preserving we can write this affine transformation as $T(x,y) = (\beta,0) + B(x,y)$ where $B = \begin{pmatrix} 1 & \alpha-\beta \\ 0 & 1 \end{pmatrix}$. Since $\det(B) = 1$ we see that T is area preserving.

DEFINITION. An ϵ-chain (for $T : A \to A$) from (x,y) to (w,z) is a sequence of points

$$(x,y) = (x_0,y_0), (x_1,y_1), \ldots, (x_n,y_n) = (w,z) \in A$$

such that

$$d(T(x_i,y_i),(x_{i+1},y_{i+1})) < \epsilon \text{ for } i = 0,\ldots,n-1.$$

We can use the same notation for ϵ-chains for the lift $\hat{T} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$.

These are finite versions of the pseudo-orbits introduced in chapter 5.

LEMMA 13.1. *If $(x,y) = (x_0,y_0), (x_1,y_1), \ldots, (x_n,y_n) = (w,z)$ is an ϵ-chain from (x,y) to (w,z) and $(w,z) = (w_0,z_0), (w_1,z_1), \ldots, (w_n,z_n) = (u,v)$ is an ϵ-chain from (w,z) to (u,v) then defining $(x_{n+i},y_{n+i}) = (w_i,z_i)$ for $0 \le i \le m$ makes $(x_0,y_0), \ldots, (x_n,y_n), (x_{n+1},y_{n+1}), \ldots, (x_{n+m},y_{n+m})$ an ϵ-chain from (x,y) to (u,v).*

PROOF. This is immediate from the definitions. ■

DEFINITION. We say that a point (x, y) is *chain recurrent* if for each $\epsilon > 0$ we can find an ϵ-pseudo-orbit from (x, y) to itself.

We say that $T : A \to A$ is *chain recurrent* if for every $(x, y), (w, z) \in A$ and every $\epsilon > 0$ we can find a finite ϵ-pseudo-orbit from (x, y) to (w, z).

LEMMA 13.2. *Let $T : A \to A$ be an area preserving homeomorphism; then*

(i) *every point $(x, y) \in A$ is chain recurrent,*
(ii) *$T : A \to A$ is chain recurrent.*

PROOF. (i) Fix $\epsilon > 0$ and then by (uniform) continuity we can choose $\frac{\epsilon}{2} > \delta > 0$ such that whenever $|(x,y)-(u,v)| < \delta$ then $|T(x,y)-T(u,v)| < \frac{\epsilon}{2}$.
Let us choose a finite cover of δ-balls

$$A \subset \cup_{i=1}^N B((z_i, w_i), \delta) \text{ where } (z_1, w_1), \ldots, (z_N, w_N) \in A.$$

Since T is area preserving we have for each $i = 1, \ldots, N$ that we can choose n_i such that $T^{-n_i} B((z_i, w_i), \delta) \cap B((z_i, w_i)\delta) \neq \emptyset$. If we choose (u_i, v_i) in this intersection then

$$\begin{cases} (x_0, y_0) = (u_i, v_i), \\ (x_1, y_1) = T(u_i, v_i), \\ \vdots \\ (x_j, y_j) = T^j(u_i, v_i), \\ \vdots \\ (x_{n_i-1}, y_{n_i-1}) = T^{(n_i-1)}(u_i, v_i), \\ (x_{n_i}, y_{n_i}) = (u_i, v_i) \end{cases}$$

gives an $\frac{\epsilon}{2}$-chain from (z_i, w_i) to itself since we observe that

(a) $|T((x_0, y_0)) - T(z_i, w_i)| < \frac{\epsilon}{2}$ (since $|(x_0, y_0) - (z_i, w_i)| = |(u_i, v_i) - (z_i, w_i)| < \delta$),
(b) $T(x_j, y_j) = T(T^j(u_i, v_j)) = T^{j+1}(u_j, v_j)$ for $j = 1, \ldots, n_i - 2$; and
(c) $T(x_{n_i-1}, y_{n_i-1}) = T^{n_i}(u_j, v_j) \in B((z_i, w_i), \delta) \subset B((z_i, w_i), \frac{\epsilon}{2}))$.

For any $(z, w) \in A$ we can choose some (z_i, w_i) $(i = 1, \ldots, N)$ which is δ-close to (z, w) (i.e. $|(z_i, w_i) - (z, w)| < \delta$). We then see that the above $\frac{\epsilon}{2}$-chain from (z_i, w_i) to itself also serves as an ϵ-chain from (z, w) to itself on replacing both (x_0, y_0) and (x_{n_i}, y_{n_i}) by (z, w). To see this observe that:

(d) $|T(z, w) - (x_1, y_1)| \leq |T(z, w)) - T(z_i, w_i)| + |T(z_i, w_i) - T(x_1, y_1)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ (since $|(z_i, w_i) - (z, w)| < \delta$ implies that $|T(z_i, w_i) - T(z, w)| < \frac{\epsilon}{2}$),
(e) $|T(x_{n_i}, y_{n_i}) - (z, w)| \leq |T(x_{n_i}, y_{n_i}) - (z_i, w_i)| + |(z_i, w_i) - (z, w)| \leq \frac{\epsilon}{2} + \delta < \epsilon$).

(ii) We may choose a sequence $(z_{i_0}, w_{i_0}), (z_{i_1}, w_{i_1}), \ldots, (z_{i_n}, w_{i_n})$ such that

$$\begin{cases} (x,y) \in B((z_{i_0}, w_{i_0}), \delta), \\ B((z_{i_j}, w_{i_j}), \delta) \cap B((z_{i_{j+1}}, w_{i_{j+1}}), \delta) \neq \emptyset \text{ for } 0 \leq j \leq n-1, \\ (u,v) \in B((z_{i_n}, w_{i_n}), \delta). \end{cases}$$

If we write in succession the ϵ-pseudo-orbit from (z_{i_0}, w_{i_0}) to (z_{i_0}, w_{i_0}), and then the $\frac{\epsilon}{2}$-pseudo-orbit from (z_{i_1}, w_{i_1}) to (z_{i_1}, w_{i_1}), etc. until we get to the $\frac{\epsilon}{2}$-pseudo-orbit from (z_{i_n}, w_{i_n}) to (z_{i_n}, w_{i_n}) then resulting concatenated sequence is an ϵ-chain from (x,y) to (u,v) (cf. Lemma 13.1).

■

The following result tells us that there is no distinction between the existence of fixed points and periodic points for the map $\hat{T} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$.

BROUWER PLANE THEOREM. *If $\hat{T} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ has a periodic point (i.e. $\exists n \neq 1$, $\hat{T}^n(x,y) = (x,y)$) then $\hat{T}$ has a fixed point (i.e. $\exists T(u,v) = (u,v)$).*

This is a classical result. We sketch the proof in the final section. (For detailed proofs we refer to [2], [1].)

THEOREM 13.3 (POINCARÉ-BIRKHOFF). *Assume that $T : A \to A$ is an area preserving homeomorphism and that the rotation numbers ρ_0 and ρ_1 for $T : \mathbb{R}/\mathbb{Z} \times \{0\} \to \mathbb{R}/\mathbb{Z} \times \{0\}$ and $T : \mathbb{R}/\mathbb{Z} \times \{0\} \to \mathbb{R}/\mathbb{Z} \times \{0\}$, respectively, satisfy either $\rho_0 < 0 < \rho_1$ or $\rho_1 < 0 < \rho_0$. Then there exists a fixed point for T.*

REMARK. This result is also known as *Poincaré's last geometric theorem.* Usually the statement involves the existence of two distinct fixed points.

Theorem 13.3 is a special case of the following more general result.

THEOREM 13.4 (FRANKS). *Assume that $T : A \to A$ is chain recurrent and that the rotation numbers ρ_0 and ρ_1 for $T : \mathbb{R}/\mathbb{Z} \times \{0\} \to \mathbb{R}/\mathbb{Z} \times \{0\}$ and $T : \mathbb{R}/\mathbb{Z} \times \{0\} \to \mathbb{R}/\mathbb{Z} \times \{0\}$, respectively, satisfy either $\rho_0 < 0 < \rho_1$ or $\rho_1 < 0 < \rho_0$. Then there exists a fixed point for T.*

PROOF OF THEOREM 13.3 (ASSUMING THEOREM 13.4). By Lemma 13.2 the hypothesis that T is area preserving implies that T is chain recurrent. The results follows immediately from Theorem 13.4.

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It remains to prove Theorem 13.4.

PROOF OF THEOREM 13.4. The proof of the theorem will be conveniently divided into the following sublemmas.

SUBLEMMA 13.4.1. *Let $\hat{T} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ be the lift of a chain recurrent homeomorphism $T : A \to A$. There are four possibilities:*

(i) $\hat{T} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ *has a chain recurrent point,*
(ii) *all points move to the right (i.e.* $\forall (x,y) \in \mathbb{R} \times [0,1]$ *if* $(x^{(n)}, y^{(n)}) := \hat{T}^n(x,y)$ *then* $\lim_{n\to+\infty} x^{(n)} = +\infty$*),*
(iii) *all points move to the left (i.e.* $\forall (x,y) \in \mathbb{R} \times [0,1]$ *we have* $\lim_{n\to+\infty} x^{(n)} = -\infty$*), or*
(iv) $\forall M > 0$, $\exists (x,y), (w,z) \in \mathbb{R} \times [0,1]$ $\exists n, m \geq 1$ *with* $(x^{(n)} - x^{(m)}) < -M$ *and* $(w^{(n)} - w) > M$.

The following result shows that case (iv) is actually redundant.

SUBLEMMA 13.4.2. *Case (iv) implies case (i).*

SUBLEMMA 13.4.3. *Let $\hat{T} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ be a lift of $T : A \to A$. Assume that $\exists (x,y) \in \mathbb{R} \times [0,1]$, $\forall \epsilon > 0$, $\exists (x_i, y_i)$, $i = 0, \dots, n$, such that $(x_0, y_0) = (x_n, y_n) = (x,y)$ and $|\hat{T}(x_i, y_i) - (x_{i+1}, y_{i+1})| < \epsilon$ for $i = 0, \dots, n-1$. Then there exists a fixed point for $\hat{T} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$.*

Assuming these sublemmas the proof of Theorem 13.4 is now a simple matter. By the area preserving hypothesis $T : A \to A$ is chain recurrent and Sublemma 13.4.1 applies.

By the hypotheses on the rotation numbers of the boundaries, points on the two boundaries move in opposite directions. Thus we see that cases (ii) and (iii) are eliminated. If (i) holds then T has a chain recurrent point. If (iv) holds than by Sublemma 13.4.2 this again leads to the same conclusion, that there exists a chain recurrent point.

Finally, by Sublemma 13.4.3 the existence of a chain recurrent point implies the existence of a fixed point.

■

We are only left with the chore of proving the sublemmas.

PROOF OF SUBLEMMA 13.4.1. Let us assume that (iv) fails, then to prove the sublemma we need to show that either (i), (ii) or (iii) holds.

Let us assume that (iv) fails because $\exists M > 0$, $\forall (x,y) \in \mathbb{R} \times [0,1]$, $\forall n \geq 1$ we have that $x^{(n)} - x \geq -M$. We have two possibilities.

Firstly, if for some $(x,y) \in \mathbb{R} \times [0,1]$ we have that the sequence $x^{(n)} - x$, $n \geq 1$, is also bounded above then the sequence $(\hat{T}^i(x,y))_{i=0}^{\infty}$ is confined to a bounded region of $\mathbb{R} \times [0,1]$ and so must have an accumulation point (x^*, y^*), say. However, for any $\delta > 0$ we need only choose $n' > n \geq 1$ with $|\hat{T}^{n'}(x,y) - (x^*, y^*)| < \frac{\delta}{2}$ and $|\hat{T}^n(x,y) - (x^*, y^*)| < \frac{\delta}{2}$ and then the sequence $\left((x^{(i)}, y^{(i)})\right)_{i=n}^{n'}$ is a δ-chain from (x^*, y^*) back to (x^*, y^*). Thus (x^*, y^*) is a chain recurrent point and we are in case (i).

The second possibility is that $\forall (x,y) \in \mathbb{R} \times [0,1]$ the sequence $x^{(n)} - x \geq -M$ $(n \geq 0)$ is unbounded (i.e. $\hat{T}^n$ may move points arbitrarily far to the right, but never to the left). In particular, for any $C > 0$ we can choose N with $x^{(N)} - x \geq C$. Thus if $n \geq N$ then $x^{(n)} - x = \left(x^{(N)}\right)^{(n-N)} - x^{(N)} + (x^{(N)} - x) \geq -M + C$. i.e. $\lim_{n\to+\infty} x^{(n)} = +\infty$ (thus we are in case (ii)).

If we had assumed that (iv) failed because the second condition in (iv) was not met then a similar argument would have given that we are in either case (i) or case (iii).

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PROOF OF SUBLEMMA 13.4.2. We need a preliminary observation. Consider any two points $(u,v),(s,t) \in A$; then by Lemma 13.2 (i) we can find an ϵ-chain $(u_i, v_i)_{i=0}^n$ from (u,v) to (s,t). Lifting this chain to $\mathbb{R} \times [0,1]$ we get that there is an ϵ-chain in $\mathbb{R} \times [0,1]$ from (u,v) to $(s+r,t)$, say, for some $r \in \mathbb{Z}$. In addition, n can be bounded above by a bound D, say, depending only on ϵ and not on the choice of (u,v) and (s,t).

Returning to the proof of Sublemma 13.4.2, assuming property (iv) let us take $M > 4D$, then let $(x,y),(w,z) \in \mathbb{R} \times [0,1]$ be the two points described in its statement.

(a) Given any point $(u,v) \in \mathbb{R} \times [0,1]$ we can construct an ϵ-chain from (u,v) to $(x+r_1, y)$, for some $r_1 \in \mathbb{Z}$, by the above observation (with $|r_1| \leq D$).
(b) We can construct an ϵ-chain in $\mathbb{R} \times [0,1]$ from $(x+r_1,y)$ to $(x^{(n)} + r_2, y^{(n)})$, for some $r_2 \in \mathbb{Z}$, by taking the lift of the orbit sequence $\left(x^{(i)}, y^{(i)}\right)_{i=0}^n$. By hypotheses, $r_2 \geq 4D$.
(c) We can construct an ϵ-chain in $\mathbb{R} \times [0,1]$ from $(x^{(n)} + r_2, y^{(n)})$ to $(u+r_2, v)$, for some $r \in \mathbb{Z}$, by the above observation (with $|r_2| \leq D$).

Thus by Lemma 13.1 we can concatenate these to get an ϵ-chain from (u,v) to $(u,v)+(r,0)$ with $r > 2D$

A similar argument (using the second part of property (iv)) shows that there is an ϵ-chain from z to $z-(s,0)$, say, for some $s \in \mathbb{Z}$.

If $s = r$ then we can use Lemma 13.1 to combine the ϵ-chain from z to $z+(r,0)$ with the ϵ-chain from $z+(r,0)$ to $z+(r-s,0) = z$ to get an ϵ-chain from z to itself. If $r \neq s$, we can repeat s times the ϵ-chain (applying Lemma 13.1 repeatedly) from z to $z+(r,0)$ (to get an ϵ-chain from z to $z+(rs,0)$) followed by r times the ϵ-chain from z to $z-(s,0)$ (applying Lemma 13.1 repeatedly) to get from $z+(rs,0)$ to z.

■

PROOF OF SUBLEMMA 13.4.3. We shall first show that for each $\delta > 0$ we can find a homeomorphism $S : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ such that S has a fixed point and $\sup_{(x,y)\in\mathbb{R}\times[0,1]} |\hat{T}(x,y) - S(x,y)| < \frac{\delta}{2}$.

By hypothesis, we can choose a $\frac{\delta}{4}$-pseudo-orbit (x_i, y_i), $i = 0, \ldots, N$, from (x,y) to itself. We introduce a homeomorphism $h : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ such that $h(\hat{T}(x_i,y_i)) = (x_{i+1}, y_{i+1})$ and $\sup_{(x,y)\in\mathbb{R}\times[0,1]} |h(x,y) - (x,y)| < \frac{\delta}{2}$.

(Intuitively, this seems easy, although in practice it is harder to write down details.)

If we define $g(x,y) = (h \circ \hat{T})(x,y)$ then we can arrange that

$$\sup_{(x,y)\in\mathbb{R}\times[0,1]} |\hat{T}(x,y) - S(x,y)| < \frac{\delta}{2} \text{ and } S^n(x,y) = (x,y).$$

Thus there exists a periodic point for S and therefore by Brouwer's theorem there is a fixed point for S.

We observe that if we assume for a contradiction that $\hat{T}$ did *not* have a fixed point then (by compactness of A) the same would be true for any sufficiently close homeomorphim S. This contradicts the above construction.

Finally, this fixed point for $\hat{T}$ projects to a fixed point for $T : A \to A$.

13.2 Outline proof of Brouwer's theorem

In the previous section we made use of a classical (but not standard) result of Brouwer. In this section we shall ouline the main ideas in the proof.

OUTLINE PROOF. The proof has two distinct parts:

(i) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism with a periodic point of prime period $n \geq 3$ then there exists a homeomorphism $T' : \mathbb{R}^2 \to \mathbb{R}^2$ with either a fixed point or a periodic point of prime period at most 2 *and the two homeomorphisms have the same set of fixed points.*

(ii) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ has a periodic point of prime period 2 then there exists a fixed point.

Part (i) is proved by an iterative method. Specifically, If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism with a periodic point $T^n x = x$ of prime period $n \geq 3$ then one shows there exists a homeomorphism $T' : \mathbb{R}^2 \to \mathbb{R}^2$ with a periodic point of prime period at most $n-1$ and the two homeomorphisms have the same set of fixed points.

To see this, consider the family of balls $B(x,\epsilon)$ about x, and their images $T(B(x,\epsilon))$ as neighbourhoods of Tx (cf. Figure 13.1). We choose the *smallest* $\epsilon > 0$ such that $\mathrm{cl}(B(x,\epsilon)) \cap T(\mathrm{cl}(B(x,\epsilon))) \neq \emptyset$. We can choose a point $z \in \mathrm{cl}(B(x,\epsilon)) \cap T(\mathrm{cl}(B(x,\epsilon)))$ and a path γ in $B(x,\epsilon)$ from $T^{-1}z$ to z (passing through x). By construction, this path γ has the property that $\gamma \cap T(\gamma) = \emptyset$. Since $x \in \gamma$ we see that $T^n(\gamma) \cap \gamma \neq \emptyset$ and so the "first" intersection $y \in T^k(\gamma) \cap \gamma$ ($n \geq k \geq 2$) gives rise to a simple closed curve containing $T(\gamma) \cup T^2(\gamma) \cup \ldots \cup T^{n-1}(\gamma)$. The homeomorphism T can be changed in a continuous way, or *isotopied* (but only in a small neighbourhood of this closed curve), to $T' : \mathbb{R}^2 \to \mathbb{R}^2$ so that z becomes a point of period $k-1$ for T'.

Furthermore, since T can have no fixed points on the simple closed curve then it has some neighbourhood U in which the same is true. If we arrange

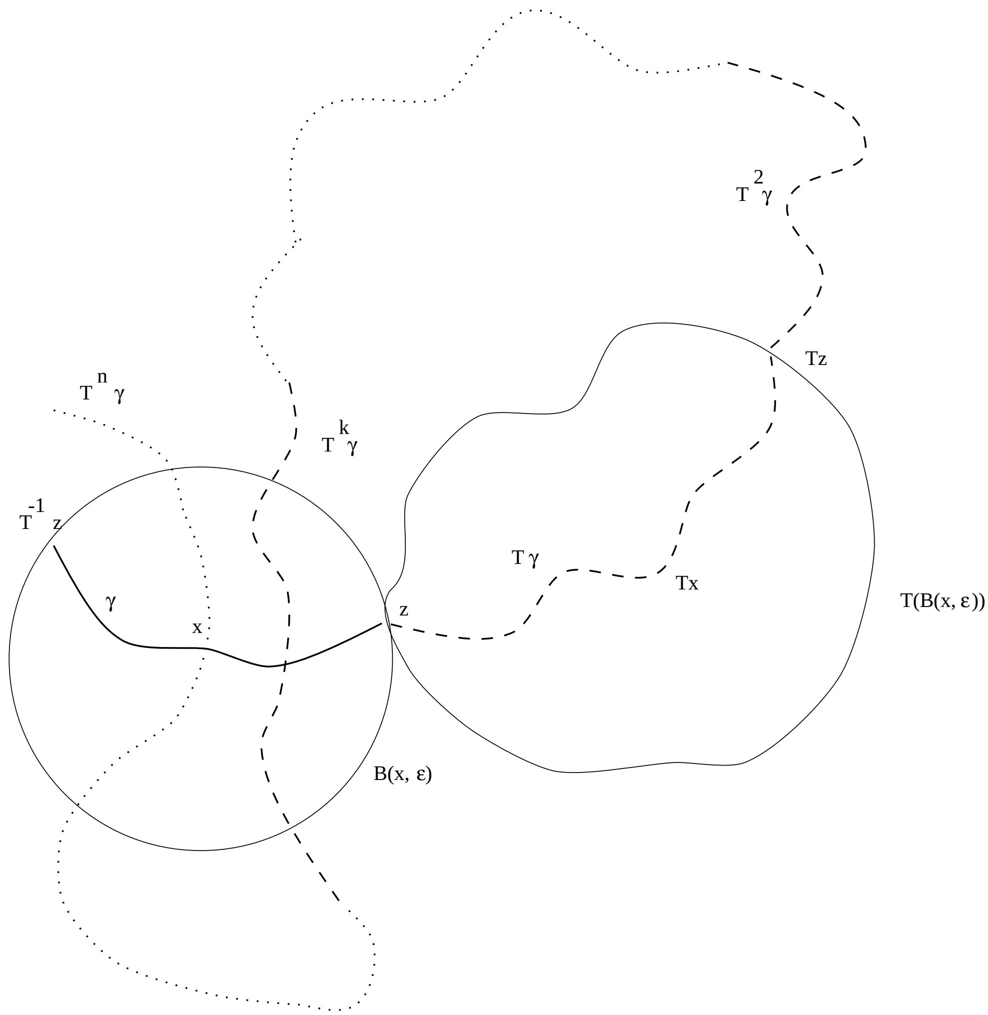

FIGURE 13.1. The proof of Brouwer's theorem

that T' differs from T only in this neighbourhood, then they have the same set of fixed points in $\mathbb{R}^2$.

Part (ii) is proved using some elementary topology. Assume that $T : \mathbb{R}^2 \to \mathbb{R}^2$ has a periodic point $T^2x = x$ (with x and Tx distinct). By "adding the fixed point at infinity" ∞ this corresponds to a homeomorphism on the standard 2-sphere S^2. By "blowing-up" the points x and Tx into circles we finally get a corresponding homeomorphism $\hat{T} : A \to A$ on a closed annulus $A = S^1 \times [0, 1]$, say, which preserves orientation (since $T : \mathbb{R}^2 \to \mathbb{R}^2$ preserved orientation) and interchanges the two boundary components (since T interchanged x and Tx). Finally, the universal cover of the annulus is $X = \mathbb{R} \times [0, 1]$, and the lift $\tilde{T} : X \to X$ interchanges the two sides $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. Let $\pi : X \to A$ be the covering projection and let $g : X \to X$ be a generator for the covering group (isomophic to $\mathbb{Z}$).

A simple application of the familiar Brouwer fixed point theorem gives that there exist fixed points $\tilde{T}z_0 = z_0$ for $\tilde{T} : X \to X$ and $(\tilde{T}g)z_1 = z_1$ for $(\tilde{T}g) : X \to X$. There are two possibilities: either $\pi(z_0) \neq \infty$ or $\pi(z_0) = \infty$. In the first case, $\pi(z)$ corresponds to a genuine fixed point for the original map $T : \mathbb{R}^2 \to \mathbb{R}^2$, and the proof is complete. In the second case, we have that $\pi(z_1)$ is distinct from $\pi(z_0) = \infty$ and so it corresponds to a genuine fixed point for the original map $T : \mathbb{R}^2 \to \mathbb{R}^2$.

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13.3 Comments and references

The Poincaré-Birkhoff theorem usual guarantees the existence of two fixed points. However, the theorem of Franks has much weaker hypotheses [3].

Modern proofs of the Brouwer plane translation theorem can be found in [1] and [2].

References

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